Maxwell's Equations (Yet Again)

Now let's go back and again examine **Maxwell's Equations**, which we first looked at in Chapter 3:

$$\nabla \times \mathbf{B}(\bar{r},t) = \mu_0 \mathbf{J}(\bar{r},t) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\bar{r},t)}{\partial t}$$

$$\nabla \times \mathbf{E}(\bar{r},t) = -\frac{\partial \mathbf{B}(\bar{r},t)}{\partial t}$$

$$\nabla \cdot \mathbf{E}(\bar{r},t) = \frac{\rho_v(\bar{r},t)}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B}(\bar{r},t) = 0$$

Now that we have introduced the concept of **dielectrics** and **magnetic material**, we can write these equations more generally as:

$$\nabla \times \mathbf{H}(\bar{r},t) = \mathbf{J}(\bar{r},t) + \frac{\partial \mathbf{D}(\bar{r},t)}{\partial t}$$

$$\nabla \times \mathbf{E}(\bar{r},t) = -\frac{\partial \mathbf{B}(\bar{r},t)}{\partial t}$$

$$\nabla \cdot \mathbf{D}(\bar{r},t) = \rho_v(\bar{r},t)$$

$$\nabla \cdot \mathbf{B}(\bar{r},t) = 0$$

These are the point form of Maxwell's Equations; we can also write them in **integral form**:

$$\oint_C \mathbf{H}(\bar{r},t) \cdot \overline{d\ell} = I_{enc} + \iint_S \frac{\partial \mathbf{D}(\bar{r},t)}{\partial t} \cdot \overline{ds}$$

$$\oint_C \mathbf{E}(\bar{r},t) \cdot \overline{d\ell} = -\iint_S \frac{\partial \mathbf{B}(\bar{r},t)}{\partial t} \cdot \overline{ds}$$

$$\oint\!\!\!\oint_S \mathbf{D}(\bar{r},t) \cdot \overline{ds} = Q_{enc}$$

$$\oint\!\!\!\oint_S \mathbf{B}(\bar{r},t) \cdot \overline{ds} = 0$$

But, we have a **problem**! Maxwell's Equations now (i.e, in material) has too **many** unknowns and too **few** equations!

To complete our electromagnetic knowledge, we must consider the constitutive equations, which are dependent on the **material properties**:

$$\mathbf{D}(\bar{r}) = \varepsilon\,\mathbf{E}(\bar{r})$$

$$\mathbf{B}(\bar{r}) = \mu\,\mathbf{H}(\bar{r})$$

$$\mathbf{J}(\bar{r}) = \sigma\,\mathbf{E}(\bar{r})$$

Now, let's consider again Maxwell's equations:

Fields

$$\nabla \times \mathbf{H}(\bar{r},t) = \mathbf{J}(\bar{r},t) + \frac{\partial \mathbf{D}(\bar{r},t)}{\partial t}$$

$$\nabla \times \mathbf{E}(\bar{r},t) = -\frac{\partial \mathbf{B}(\bar{r},t)}{\partial t}$$

$$\nabla \cdot \mathbf{D}(\bar{r},t) = \rho_v(\bar{r},t)$$

$$\nabla \cdot \mathbf{B}(\bar{r},t) = 0$$

Sources

We can interpret these equations as relating **sources** and the **fields** these sources create. The **sources** appear on **right** side of Maxwell's equations, whereas the **fields** appear on the **left**.

For example, we know that an electric field and electric flux density is created from **charge**:

$$\nabla \cdot \mathbf{D}(\bar{r},t) = \rho_v(\bar{r},t)$$

$$\mathbf{D}(\bar{r}) = \varepsilon \mathbf{E}(\bar{r})$$

But, we also know that an electric field and electric flux density can be created (induced) by a time varying **magnetic flux density**:

$$\nabla x \mathbf{E}(\bar{r},t) = -\frac{\partial \mathbf{B}(\bar{r},t)}{\partial t}$$

$$\mathbf{D}(\bar{r}) = \varepsilon \mathbf{E}(\bar{r})$$

Likewise, we know that **current** is the source of a magnetic field and magnetic flux density:

$$\nabla x \mathbf{H}(\bar{r},t) = \mathbf{J}(\bar{r},t)$$

$$\mathbf{B}(\bar{r}) = \mu \mathbf{H}(\bar{r})$$

But, note we have **one source left**! Note that it appears that a time-varying **electric** flux density can "induce" a magnetic field, much in the same way that a time-varying magnetic flux density induces and electric field.

$$\nabla x \mathbf{H}(\bar{r},t) = \frac{\partial \mathbf{D}(\bar{r},t)}{\partial t}$$

$$\mathbf{B}(\bar{r}) = \mu \mathbf{H}(\bar{r})$$

Q: *What the heck is* $\frac{\partial \mathbf{D}(\bar{r},t)}{\partial t}$ *??*

A: Try taking the **divergence** of Ampere's Law.

$$\nabla \cdot \nabla x \mathbf{H}(\bar{r},t) = \nabla \cdot \mathbf{J}(\bar{r},t) + \frac{\partial \nabla \cdot \mathbf{D}(\bar{r},t)}{\partial t}$$

Since we know that the divergence of **every** curl is zero (i.e., $\nabla \cdot \nabla x \mathbf{H}(\bar{r},t) = 0$), we find:

$$\nabla \cdot \mathbf{J}(\bar{r},t) = -\frac{\partial \nabla \cdot \mathbf{D}(\bar{r},t)}{\partial t}$$

Recall that **often** we find that the divergence of current density $\mathbf{J}(\bar{r})$ is **zero** (i.e., $\nabla \cdot \mathbf{J}(\bar{r}) = 0$), as charge cannot be created or destroyed. The **exception** is when charge "pile up", or diminish at some point. In this case, the charge density $\rho_v(\bar{r})$ **must** change as a function of time.

Recall that this was expressed as the **continuity equation**:

$$\nabla \cdot \mathbf{J}(\bar{r}) = -\frac{\partial \rho_v(\bar{r})}{\partial t}$$

We called this type of current density—whose divergence is not zero—**displacement current** $\mathbf{J}_c(\bar{r})$.

Therefore, we can state:

$$\nabla \cdot \mathbf{J}_c(\bar{r}) = -\frac{\partial \rho_v(\bar{r})}{\partial t}$$

But, recall that $\rho_v(\bar{r}) = \nabla \cdot \mathbf{D}(\bar{r})$, therefore:

$$\nabla \cdot \mathbf{J}_c(\bar{r}) = -\frac{\partial \nabla \cdot \mathbf{D}(\bar{r})}{\partial t}$$

Or, more specifically:

$$-\frac{\partial \mathbf{D}(\bar{r})}{\partial t} = \mathbf{J}_c(\bar{r}) = \text{displacement current}$$

Therefore **Ampere's Law** can be written as:

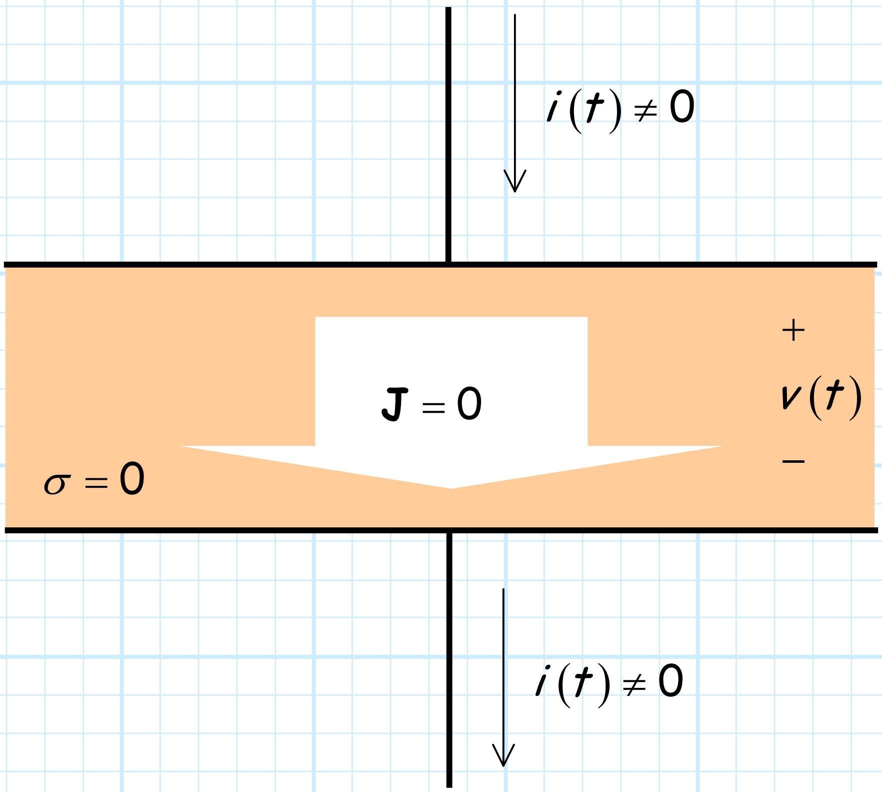
$$\begin{aligned}\nabla \times \mathbf{H}(\bar{r},t) &= \mathbf{J}(\bar{r},t) + \frac{\partial \mathbf{D}(\bar{r},t)}{\partial t} \\ &= \mathbf{J}(\bar{r},t) - \mathbf{J}_c(\bar{r},t)\end{aligned}$$

The most important **application** of displacement current is when considering **capacitors**. We know that:

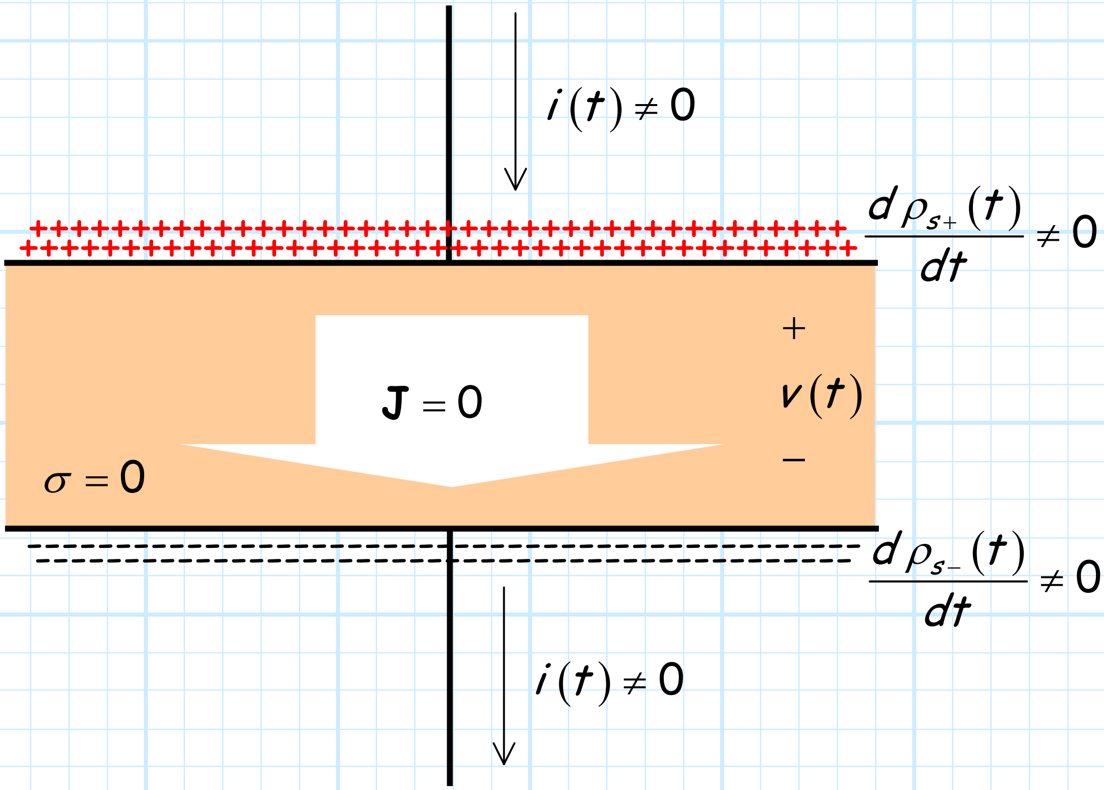
$$i(t) = C\frac{d\,v(t)}{dt}$$

Yet we also know that the conductors of a capacitor are typically separated by a dielectric with almost **no conductance** ($\sigma \approx 0$). Thus, the current density $\mathbf{J}(\bar{r})$ in the dielectric is **zero** ($\mathbf{J}(\bar{r}) = 0$).

Q: So how can current $i(t)$ be flowing ??



A: Displacement current! The charge from current $i(t)$ does not move through the capacitor, but instead "**pile up**" at each plate. This change in charge density ρ_s at each plate is equivalent to a current—a **displacement current**.



A capacitor is analogous to the "**storage tank**" that we discussed in chapter 3.